LPP graphical solution

Graphical solution to LPP

- https://www.youtube.com/watch?v=8IRrgDoV8Eo

Graphical Solution Method

1. Plot model constraint on a set of coordinates in a plane
2. Identify the feasible solution space on the graph where all constraints are satisfied simultaneously
3. Plot objective function to find the point on boundary of this space that maximizes (or minimizes) value of objective function

Maximize $Z = 50x_1 + 50x_2$

- Subject to the constraints
- $4x_1 + 3x_2 \leq 120$ lb

$x_1 + 2x_2 \leq 40$ hr

$x_1, x_2 \geq 0$

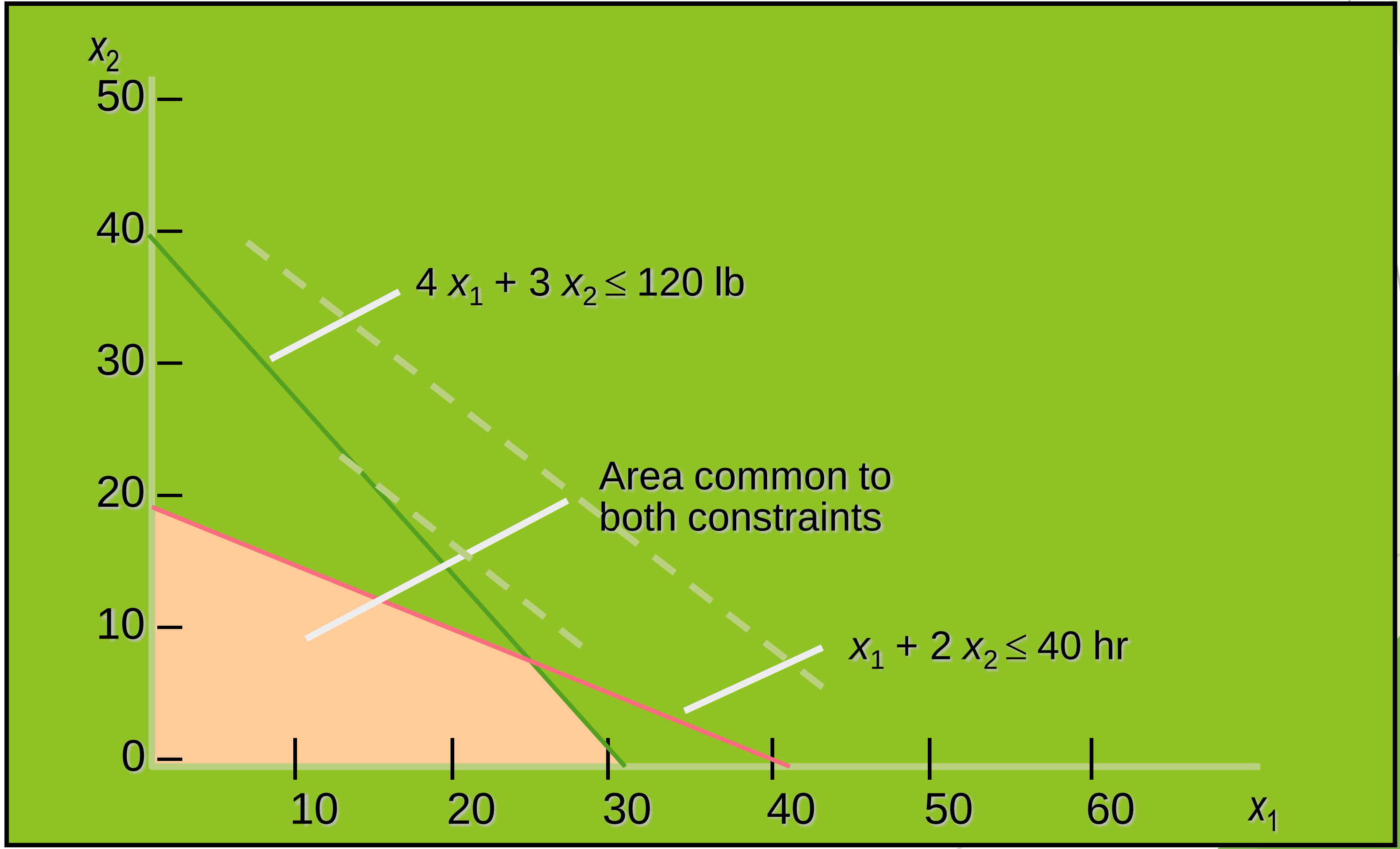
- Convert the constraints into equalities

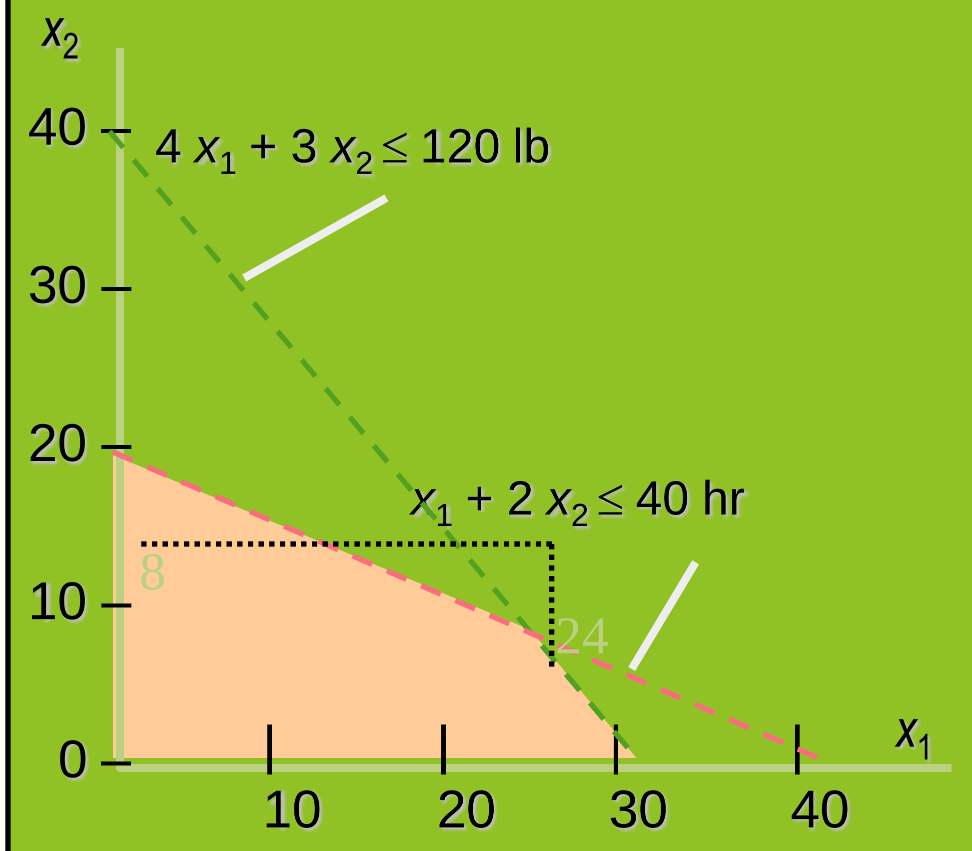
$4x_1 + 3x_2 = 120$

$x_1 + 2x_2 = 40$

- Find two coordinates for each equality; if $x_1 = 0$ then $x_2 = ?$; if $x_2 = 0$ then $x_1 = ?$;
- For $4x_1 + 3x_2 = 120$ – (0, 40) and (30,0)
- For $x_1 + 2x_2 = 40$ – (0, 20) and (40,0)

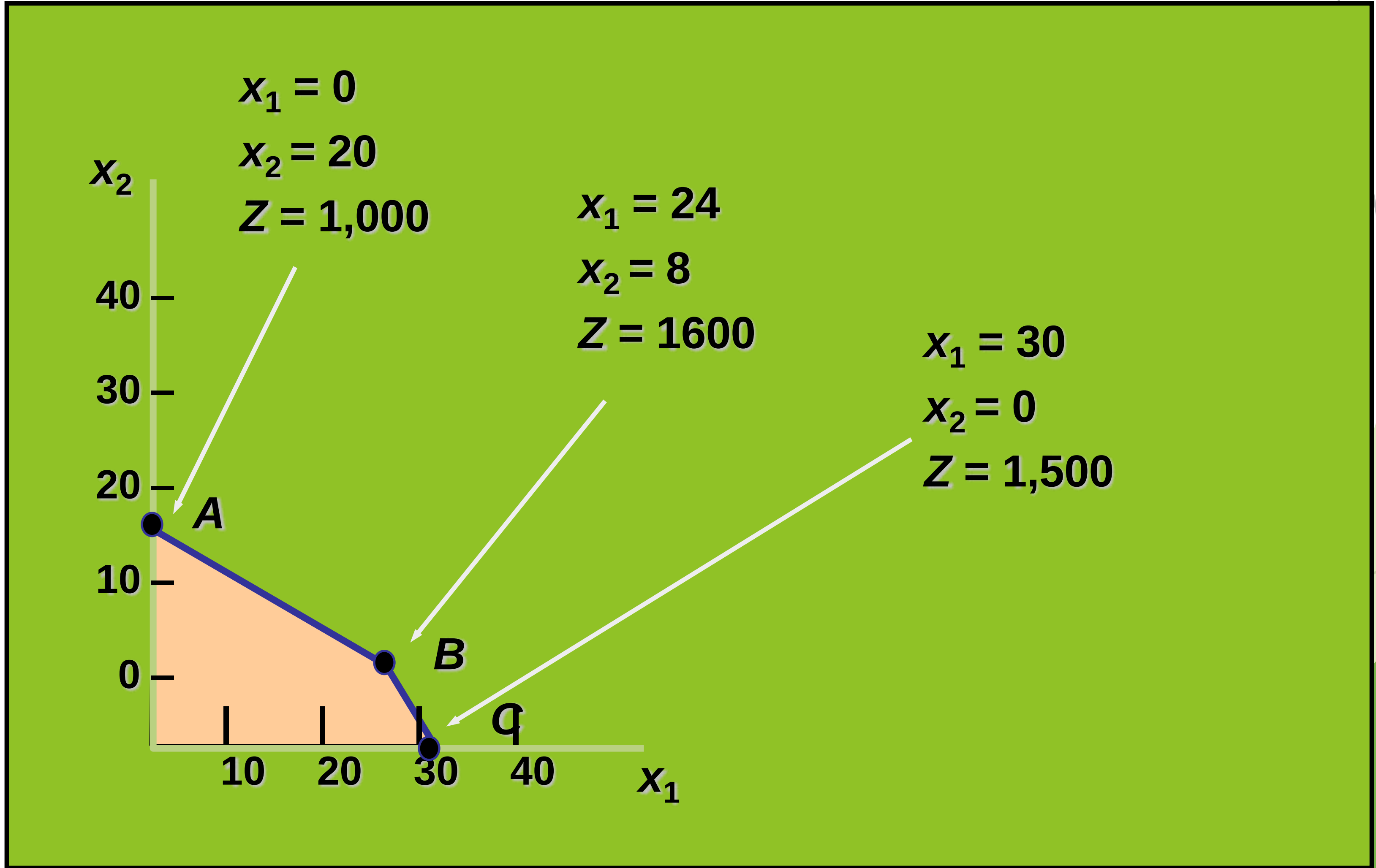
Graphical Solution: Example





$$
\begin{aligned}
x_1 + 2x_2 &= 40 \\
4x_1 + 3x_2 &= 120 \\
4x_1 + 8x_2 &= 160 \\
-4x_1 - 3x_2 &= -120 \\
5x_2 &= 40 \\
x_2 &= 8 \\
\\
x_1 + 2(8) &= 40 \\
x_1 &= 24
\end{aligned}
$$

Extreme Corner Points

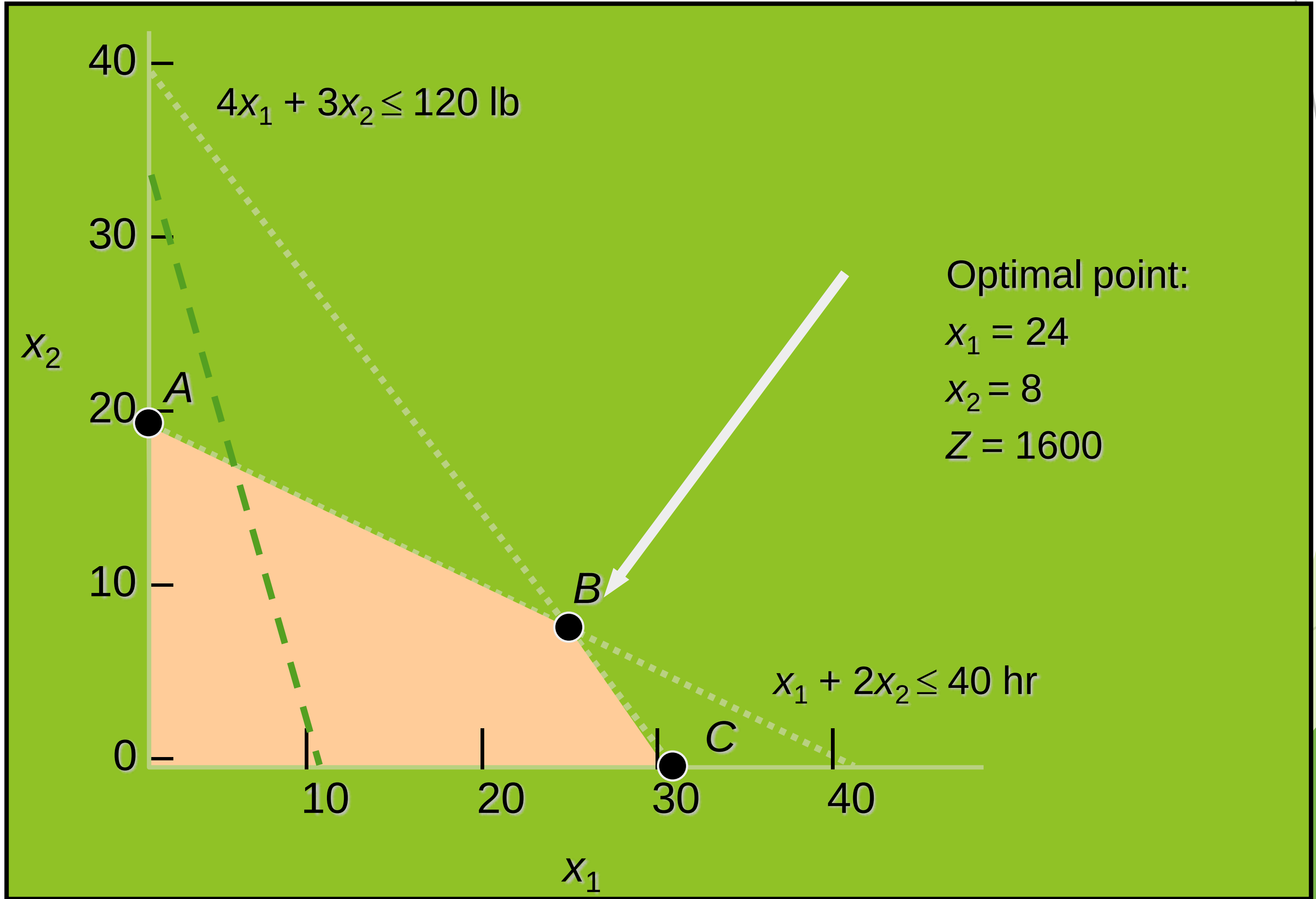


Substituting the coordinates at points A, B, C in the objective function Z = $50x_1 + 50x_2$

- At point A (0,20) → $Z = 50(0) + 50(20) = 1000$
- At point B (24,8) → $z = 50(24) + 50(8) = 1600$
- At point C (30,0) → $z = 50(30) + 50(0) = 1500$
-

- Maximum value is 1600 and corresponds to point B;
- Therefore, optimal solution is $Z = 1600$; $x_1 = 24$ and $x_2 = 8$;

Objective Function



...mple 17

Solve graphically the following LPP

Maximize $z = 5x_1 + 3x_2$

Subject to constraints $3x_1 + 5x_2 \leq 15;\ 5x_1 + 2x_2 \leq 10;\ x_1, x_2 \geq 0$

...lution :

In order to plot the constraints on the graph we convert inequalities into equations

i.e., $3x_1 + 5x_2 = 15$

$5x_1 + 2x_2 = 10$

To plot $3x_1 + 5x_2 = 15$

put $x_1 = 0,\ x_2 = 3 \Rightarrow (0, 3)$

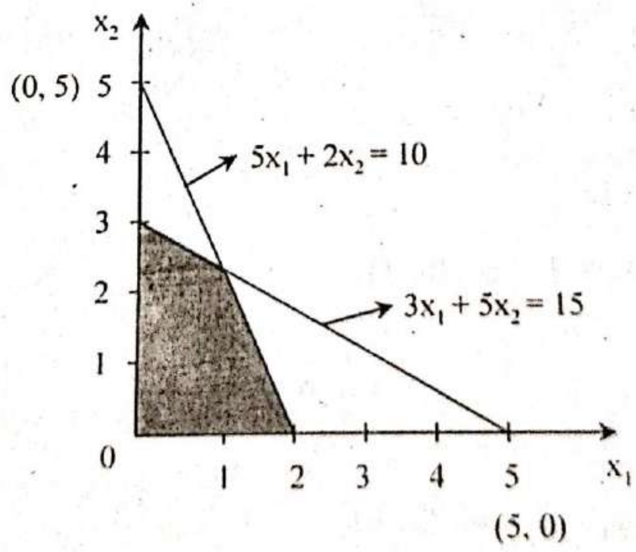
$x_2 = 0,\ x_1 = 5 \Rightarrow (5, 0)$

To plot $5x_1 + 2x_2 = 10$

put $x_1 = 0,\ x_2 = 5 \Rightarrow (0, 5)$

$x_2 = 0,\ x_1 = 2 \Rightarrow (2, 0)$

Plotting these equations on the graph, we get



The area OABC is the figure satisfied by the constraints is shown by shaded area and is
led the feasible solution region.

Corner	Coordinates of Corper Points	Max $z = 5x_1 + 3x_2$	Value
0	(0, 0)	5(0) + 3(0)	0
A	(2, 0)	5(2) + 3(0)	10
B	(1, 2.5)	5(1) + 3(2.5)	12.5
C	(0,3)	5(0) + 3(3)	9

Hence Max $z = 12.5$, the solution to the given problem is

$\therefore \quad x_1 = 1, \; x_2 = 2.5$

Example 2 :

Solve graphically the following LPP

$$\text{Maximize } Z = 3x_1 + 2z_2$$

$$\text{S.T.C.} \quad -2x_1 + x_2 \le 1$$

$$x_1 \le 2$$

$$x_1 \ge 0, \ x_2 \ge 0$$

Solution :

Convert inequalities into equations

$$-2x_1 + x_2 = 1$$

$$x_2 = 2$$

$$x_1 + x_2 = 3$$

To plot $-2x_1 + x_2 = 1$,

$$\text{put} \quad x_1 = 0, \ x_2 = 1 \Rightarrow (0, 1)$$

$$x_2 = 0, \quad x_1 = -\frac{1}{2} \Rightarrow \left(-\frac{1}{2}, 0\right)$$

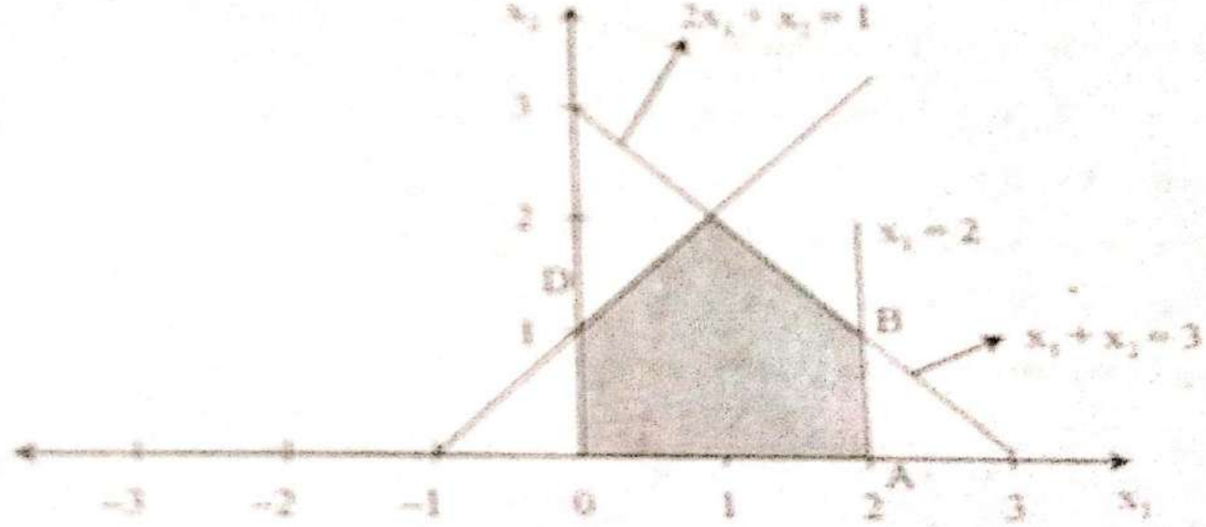
$$x_1 = 2 \Rightarrow (2, 0)$$

To plot $x_1 + x_2 = 3$,

put $x_1 = 0,\ x_2 = 3 \Rightarrow (0, 3)$

$x_2 = 0,\ x_1 = 3 \Rightarrow (3, 0)$

Plotting these equations on the graph, we get



Corner Points	Coordinites	Max $Z = 3x_1 + 2x_2$	Value
O	(0, 0)	3(0) + 2(0)	0
A	(2, 0)	3(2) + 2(0)	6
B	(2, 1)	3(2) + 2(1)	8
C	(1, 2.3)	3(1) + 2(2.3)	7.6
D	(0, 1)	3(0) + 2(1)	2

Hence Max $Z = 8$, $x_1 = 2$, $x_2 = 1$.

Minimization Problem

Two brands of fertilizer available - Super-gro, Crop-quick.
Field requires at least 16 pounds of nitrogen and 24 pounds of phosphate.
Super-gro costs $6 per bag, Crop-quick $3 per bag.
Problem : How much of each brand to purchase to minimize total cost of fertilizer given following data ?

	CHEMICAL CONTRIBUTION	
Brand	*Nitrogen (lb/bag)*	*Phosphate (lb/bag)*
Gro-plus	2	4
Crop-fast	4	3

- Minimize $Z = \$6x1 + \$3x2$
- subject to
- $2x1 + 4x2 \geq 16$ lb of nitrogen
- $4x1 + 3x2 \geq 24$ lb of phosphate
- $x1, x2 \geq 0$

Graphical method

Complete model formulation:

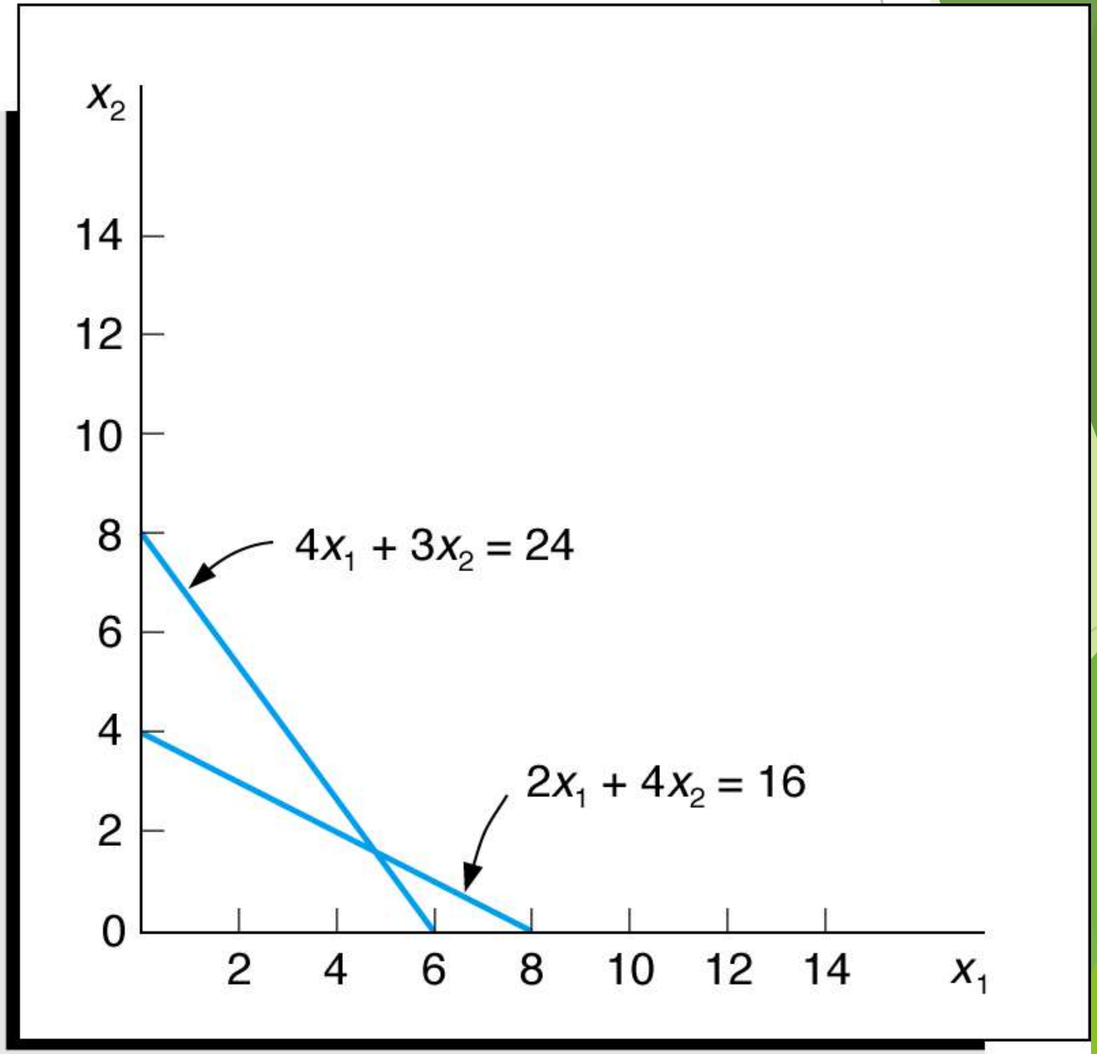
minimize $Z = \$6x_1 + 3x_2$

subject to

$2x_1 + 4x_2 \geq 16$ lb of nitrogen

$4x_1 + 3x_2 \geq 24$ lb of phosphate

$x_1, x_2 \geq 0$



A Minimization Model Example Feasible Solution Area

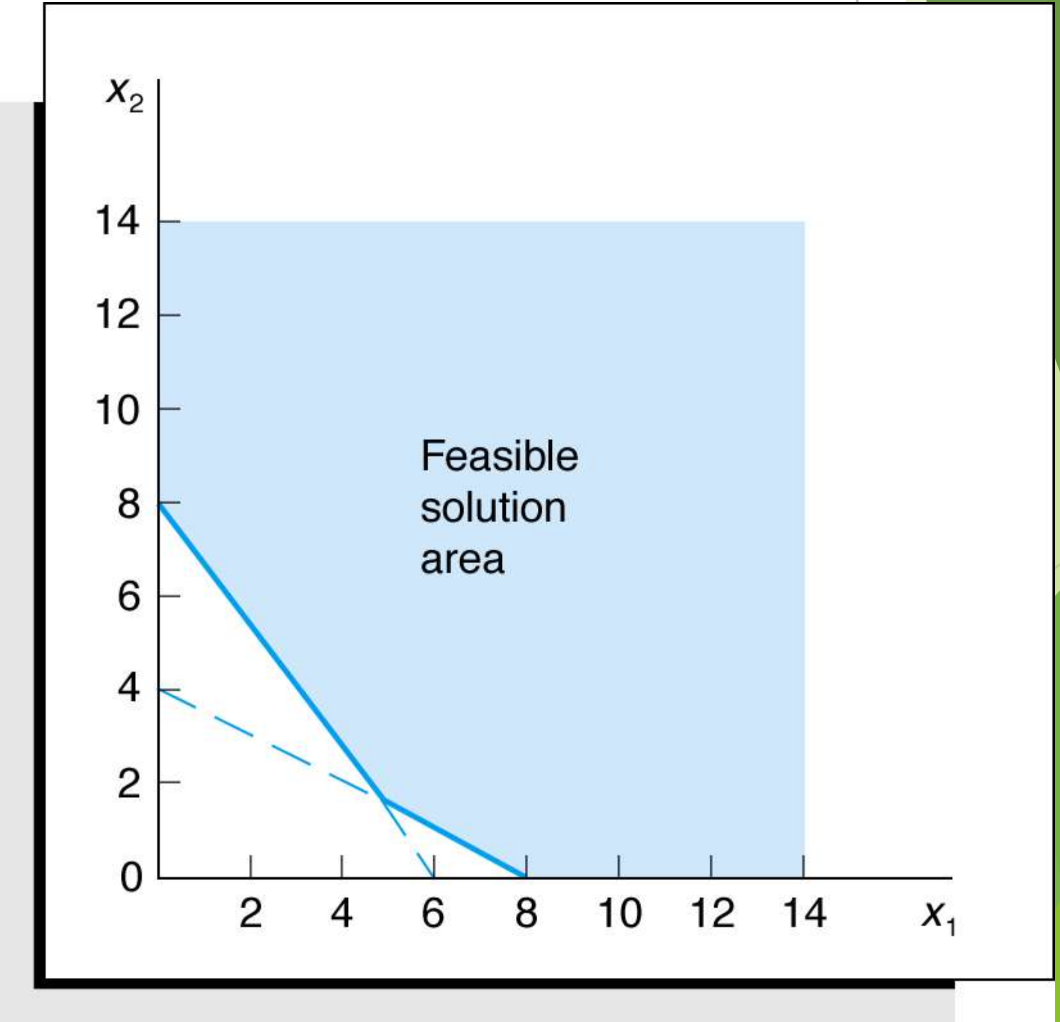
minimize $Z = \$6x_1 + 3x_2$

subject to

$2x_1 + 4x_2 \geq 16$ lb of nitrogen

$4x_1 + 3x_2 \geq 24$ lb of phosphate

$x_1, x_2 \geq 0$



A Minimization Model Example
Optimal Solution Point

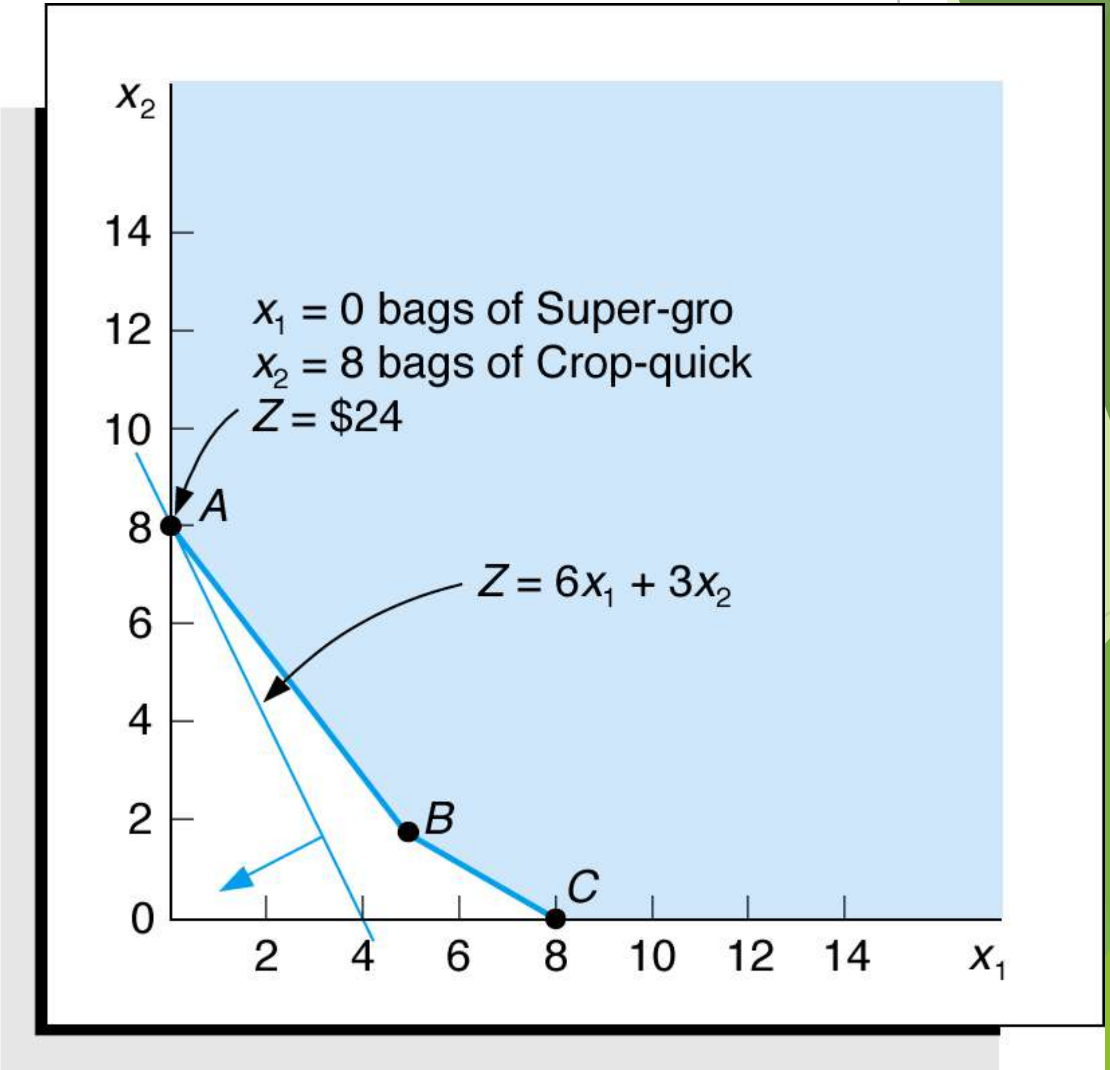
minimize $Z = \$6x_1 + 3x_2$

subject to

$2x_1 + 4x_2 \geq 16$ lb of nitrogen

$4x_1 + 3x_2 \geq 24$ lb of phosphate

$x_1, x_2 \geq 0$



Example 3 :

Solve the following LPP graphically

Min $Z = 4x_1 + 2x_2$

S.T.C. $x_1 + 2x_2 \geq 2$

$3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$; $x1, x2 \geq 0$

Solution :

Convert inequations into equations

$x_1 + 2x_2 = 2, \; 3x_1 + x_2 = 3, \; 4x_1 + 3x_2 = 6$

To plot $x_1 + 2x_2 = 2$

put $x_1 = 0,\ x_2 = 1 \Rightarrow (0, 1)$

$x_2 = 0,\ x_1 = 2 \Rightarrow (2, 0)$

To plot $3x_1 + x_2 = 3$

put $x_1 = 0,\ x_2 = 3 \Rightarrow (0, 3)$

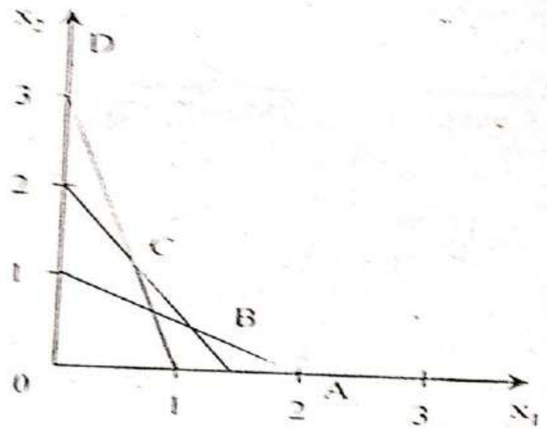
$x_2 = 0,\ x_1 = 1 \Rightarrow (1, 0)$

To plot $4x_1 + 3x_2 = 6$

put $x_1 = 0,\ x_2 = 2 \Rightarrow (0, 2)$

$x_2 = 0,\ x_1 = 1.5 \Rightarrow (1.5, 0)$

Plotting these equations in the graphs, we get



Corner Points	Coordinites	Max $Z = 4x_1 + 2x_2$	Value
A	(2, 0)	4(2) + 2(0)	8
B	(1.2, 0.4)	4(1.2) + 2(0.4)	5.6
C	(0.6, 1.2)	4(0.6) + 2(1.2)	4.8
D	(3, 0)	4(3) + 2(0)	6

The minimum value of z is 4.8 which occurs at C = (0.6, 1.2).

Hence, the solution to the above problem is $x_1 = 0.6$; $x_2 = 1.2$, min $z = 4.8$

Irregular Types of Linear Programming Problems

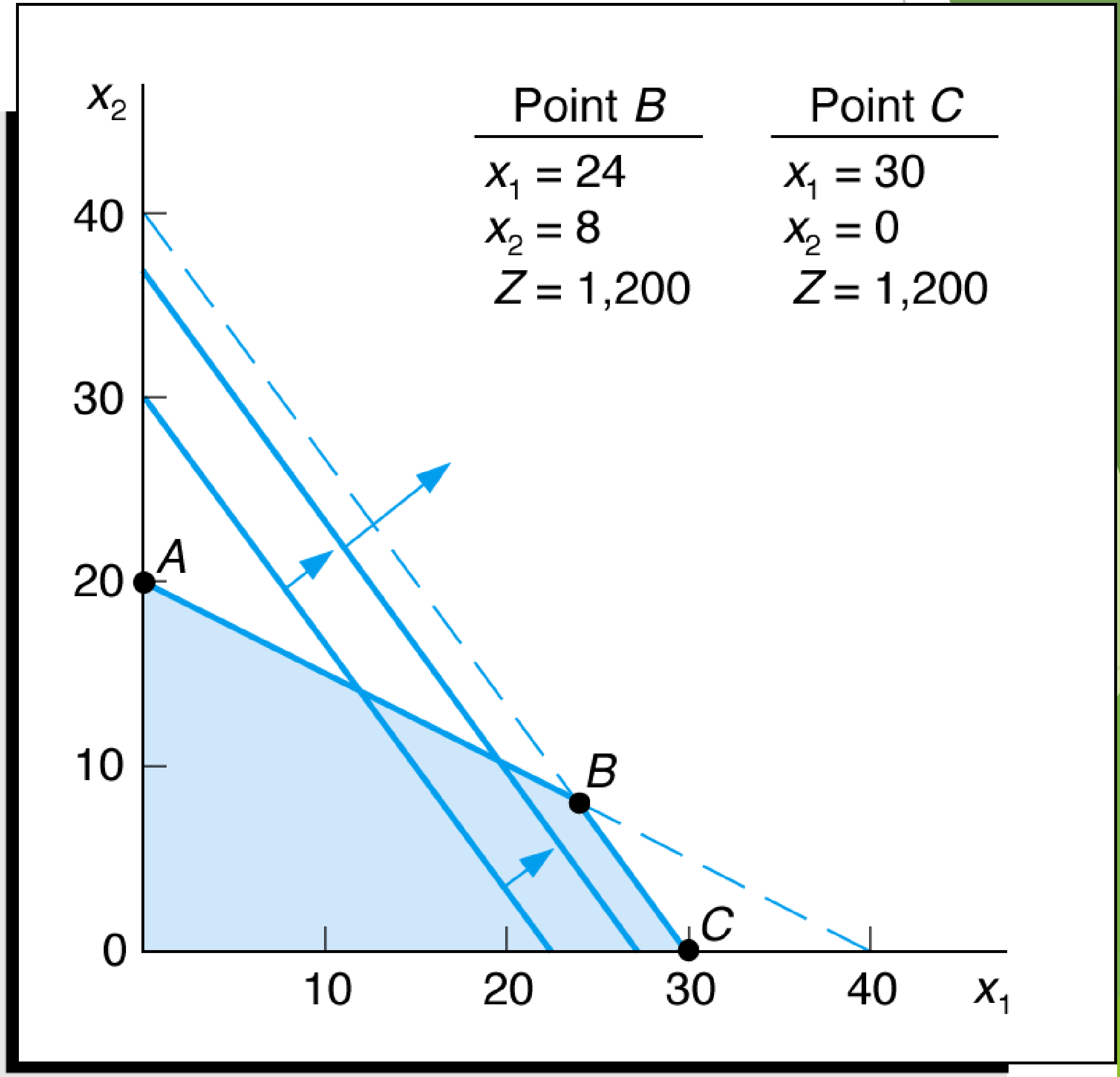
- For some linear programming models, the general rules do not apply.
- Special types of problems include those with:

1. Multiple optimal solutions
2. Infeasible solutions
3. Unbounded solutions

Multiple Optimal Solutions

Objective function is parallel to a constraint line:

maximize $Z = \$40x_1 + 30x_2$
subject to
$1x_1 + 2x_2 \leq 40$ hours of labor
$4x_2 + 3x_2 \leq 120$ pounds of clay
$x_1, x_2 \geq 0$
where x_1 = number of bowls
x_2 = number of mugs



An Infeasible Problem

Every possible solution violates at least one constraint:

maximize $Z = 5x_1 + 3x_2$

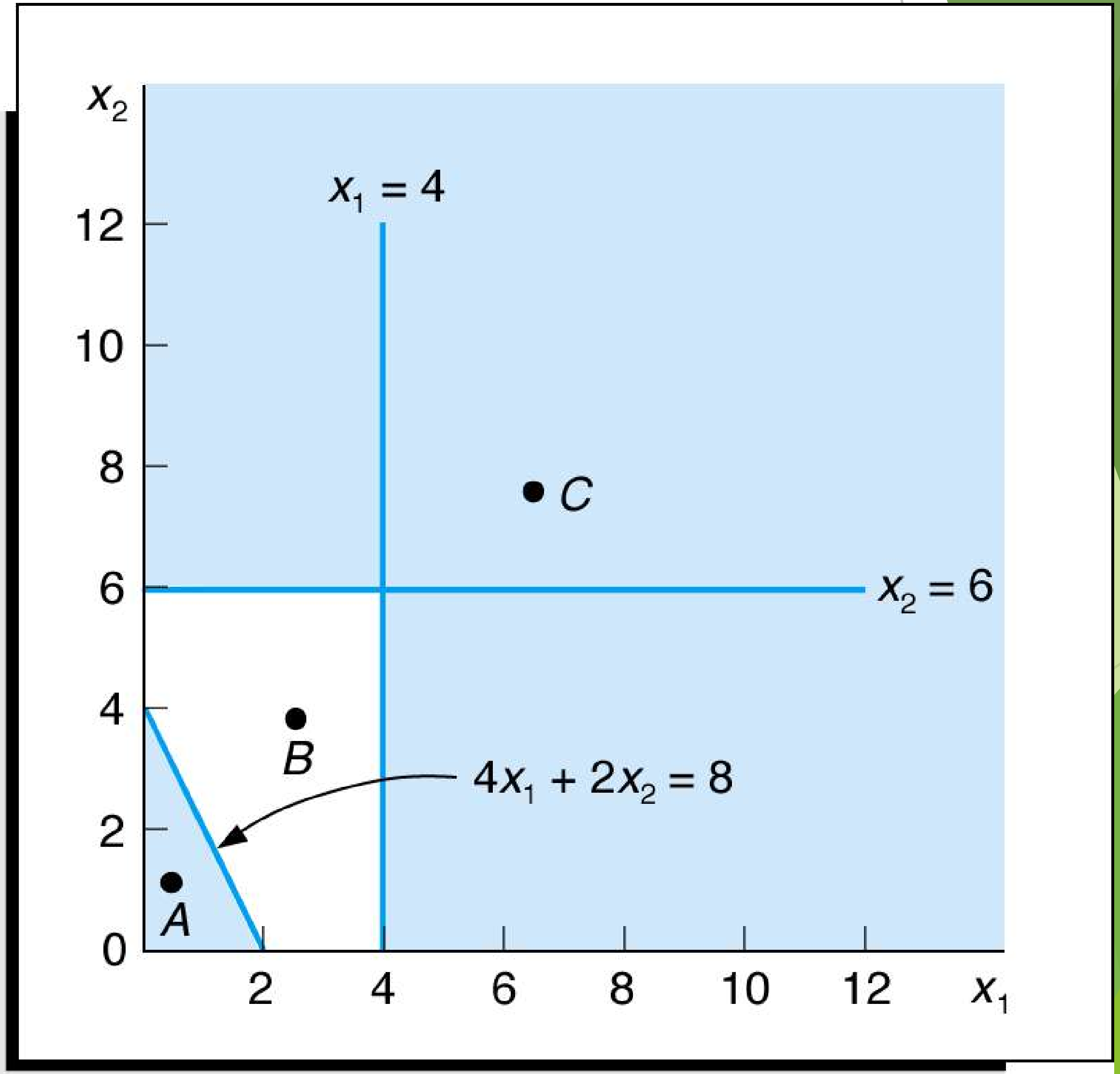
subject to

$4x_1 + 2x_2 \leq 8$

$x_1 \geq 4$

$x_2 \geq 6$

$x_1, x_2 \geq 0$

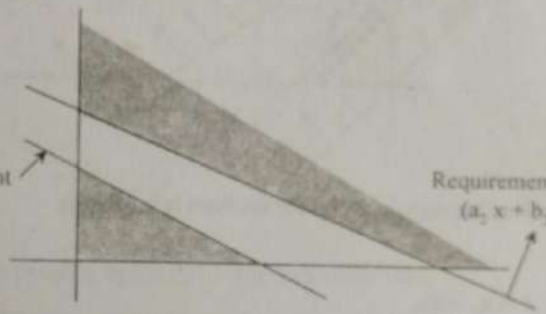


1.25 SPECIAL CASES IN GRAPHICAL APPROACH :

(a) Infeasible Solutions :

When feasible region does not exist, the solution we get is infeasible.

In graphical solution it is found when one constraint is availability ($\leq$) type and the other is requirement ($\geq$) type and these two can not produce any common area (non intersecting) in the specific quadrant (such as $x_1 \geq 0$, $x_2 \geq 0$ indicates first quadrant).



PROBLEMS ON INFEASIBLE SOLUTIONS

Example 1 :

Solve Graphically :

Maximise $Z = 50x_1 + 60x_2$

Subject to $x_1 + x_2 \leq 12$

$2x_1 + 3x_2 \geq 60$

$x_1, x_2 \geq 0$

Solution :

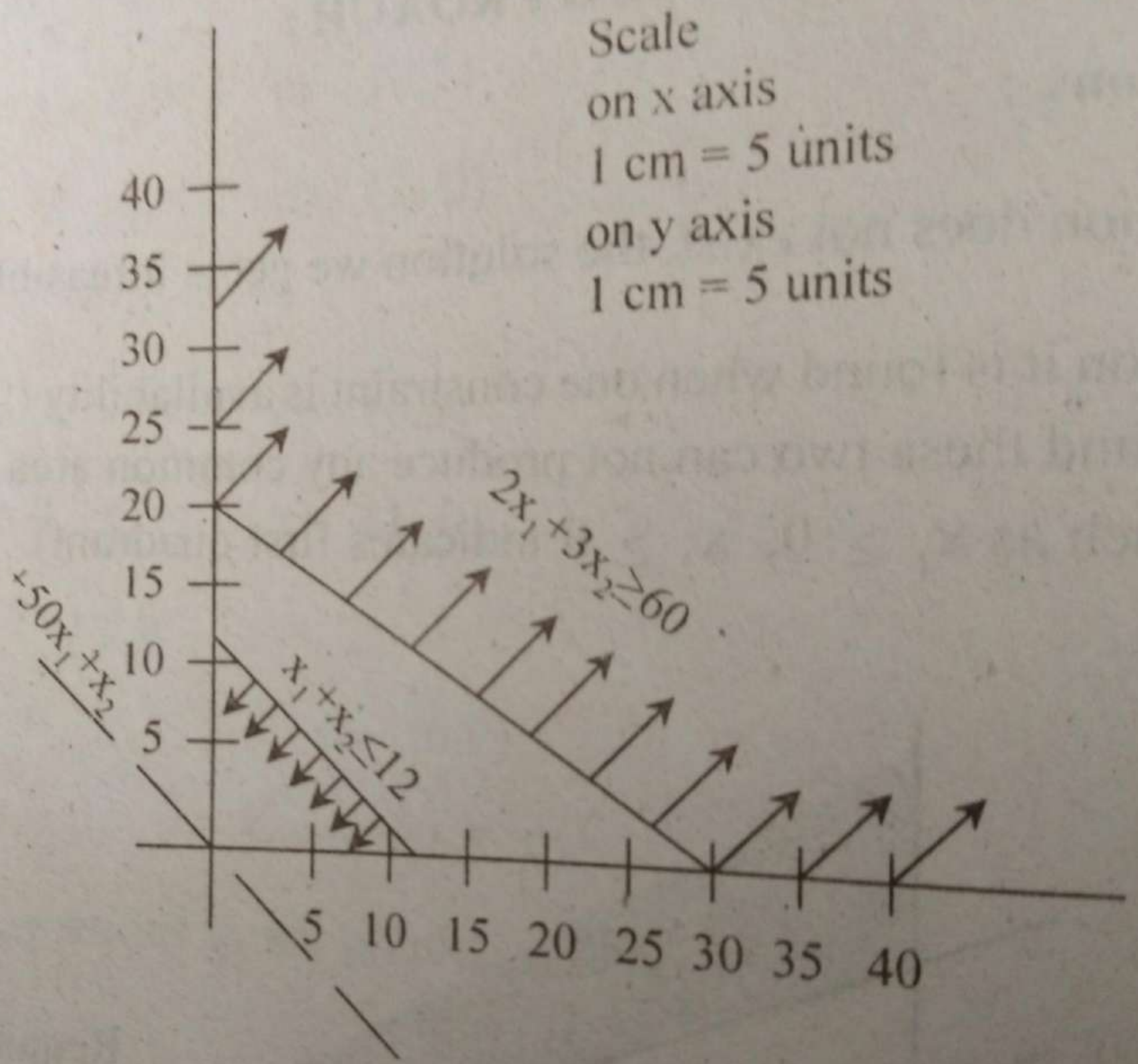
$x_1 + x_2 = 12$

x_1	0	2
x_2	12	0

$2x_1 + 3x_2 = 60$

x_1	0	30
x_2	20	0

2020.05.13 10:28



There is no feasible region and so the solution is infeasible.

mple 2 :

An Unbounded Problem

Value of objective function increases indefinitely:

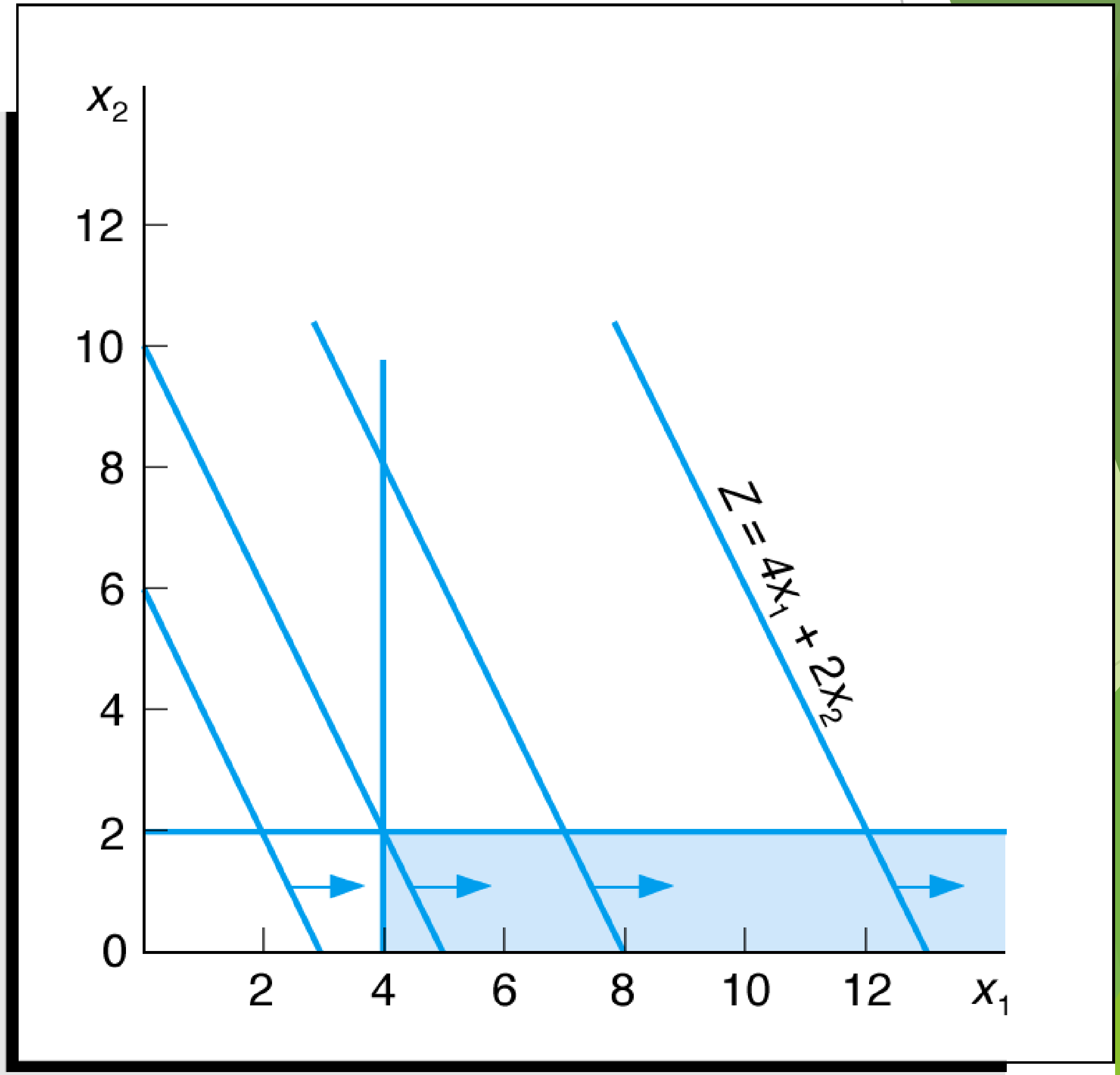
maximize $Z = 4x_1 + 2x_2$

subject to

$x_1 \geq 4$

$x_2 \leq 2$

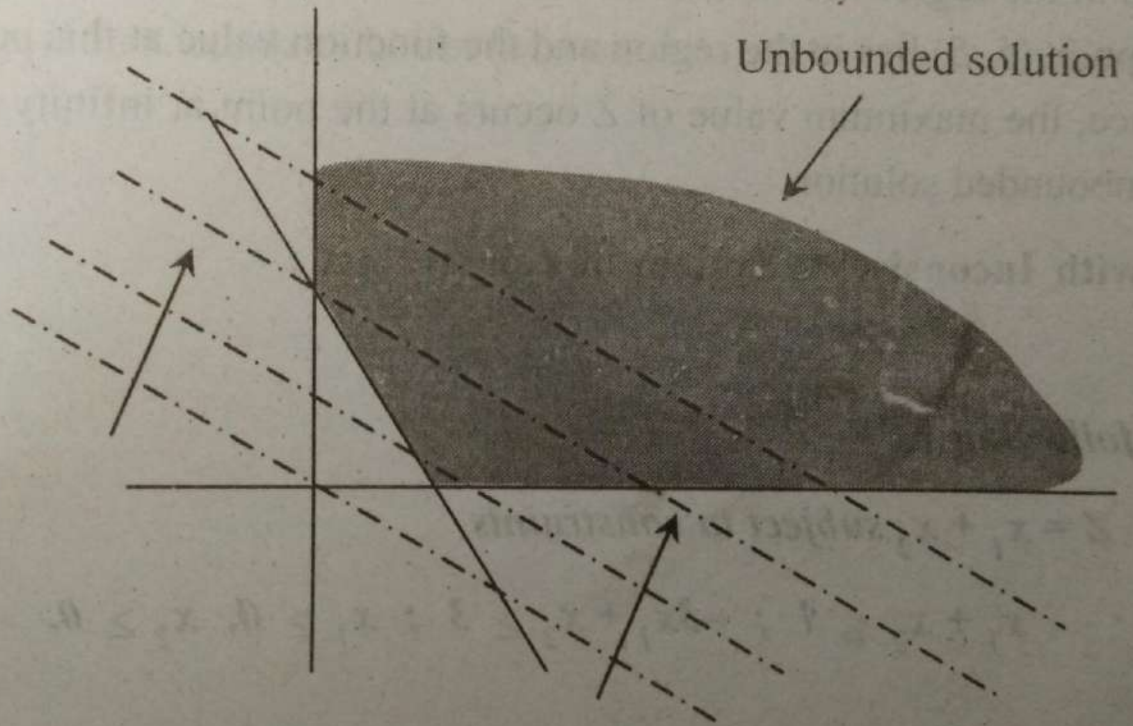
$x_1, x_2 \geq 0$



(b) **Unbounded Solution :**

If a distinct and finite solution can not be found or the solution exists at infinity, the solution is said to be unbounded.

In graphical solution, unbounded solutions are obtained if the feasible region is unbounded (formed by requirement constraints i.e., $\geq$ type) while the objective function is maximisation.



(Since it has to be taken to infinity to locate maximum value we have no finite or unbounded solution).

PROBLEMS ON UNBOUNDED SOLUTIONS

Example :

Solve the following LPP graphically

Maximize $Z = 3x_1 + 2z_2$

s.t.c. $x_1 - x_2 \leq 1$

$x_1 + x_2 \geq 3$; $x_1, x_2 \geq 0$

Solution :

Convert inequations into equations

$x_1 - x_2 = 1$ and $x_1 + x_2 = 3$

To plot $x_1 - x_2 = 1$

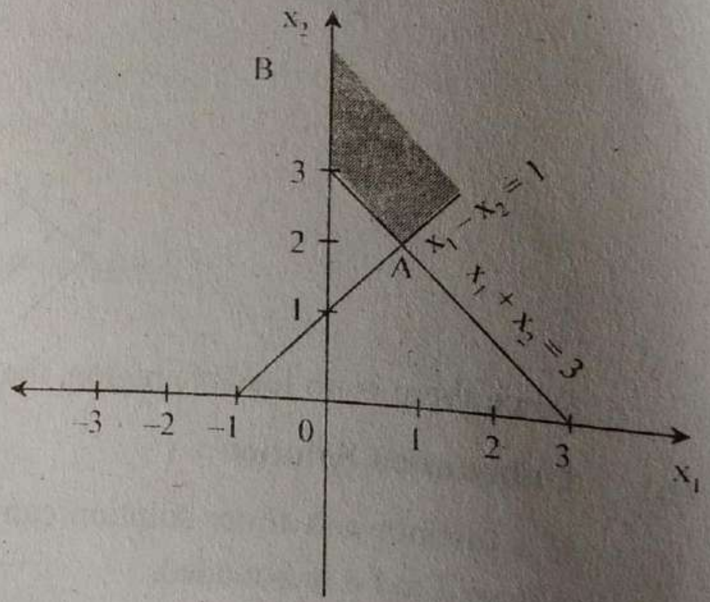
put $x_1 = 0$, $x_2 = -1 \Rightarrow (0, -1)$

$x_2 = 0$, $x_1 = 1 \Rightarrow (1, 0)$

To plot $x_1 + x_2 = 3$

put $x_1 = 0$, $x_2 = 3 \Rightarrow (0, 3)$

$x_2 = 0$, $x_1 = 3 \Rightarrow (3, 0)$



Here the shaded region is unbounded. The two vertices of the region are B = (0, 3); A = (2, 1). The values of the objective function at these vertices are Z(A) = 6 and Z(B) = 8. But there exists points in the region for which the values of the objective function is more than 8. For example, the point (5, 5) lies in the region and the function value at this point is 25 which is more than 8. Hence, the maximum value of Z occurs at the point at infinity only an

...d solution.

(c) Problem with Inconsistent System of Constraints :

Example :

Solve the following LPP

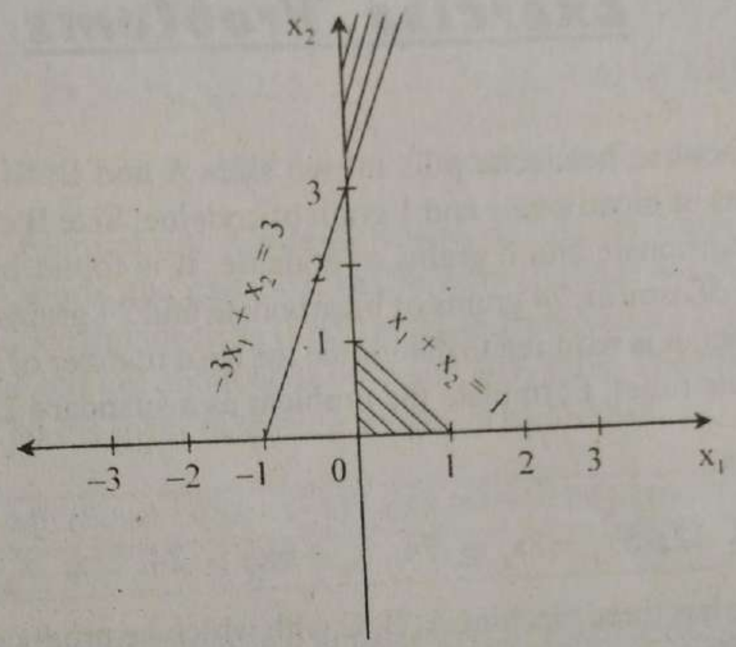
Maximize $Z = x_1 + x_2$ *subject to constraints*

$$x_1 + x_2 \le 1 \; ; \; -3x_1 + x_2 \ge 3 \; ; \; x_1 \ge 0, \; x_2 \ge 0.$$

Solution :

Consider each inequality as equation

$$x_1 + x_2 = 1 \; ; \; -3x_1 + x_2 = 3$$



To plot $x_1 + x_2 = 1,$

put $x_1 = 0,\ x_2 = 1 \Rightarrow (0, 1)$

$x_2 = 0,\ x_1 = 1 \Rightarrow (1, 0)$

To plot $-3x_1 + x_2 = 3$

put $x_1 = 0,\ x_2 = 3 \Rightarrow (0, 3)$

$x_2 = 0,\ x_1 = -1 \Rightarrow (0, -1)$

The figure shows that there is no point (x_1, x_2) which satisfies both constraints simultaneously.

Hence, the prob'em has no solution because the constraints are inconsistent.

Degeneracy in LPP

- LP is **degenerate** if in a basic feasible solution, one of the basic variables takes on a zero value
- **Degeneracy** is caused by redundant constraint(s)

Degeneracy – Special cases (cont.)

Example:

$$\textbf{Max } f = 3x_1 + 9x_2$$

Subject to:

$x_1 + 4x_2 \leq 8$

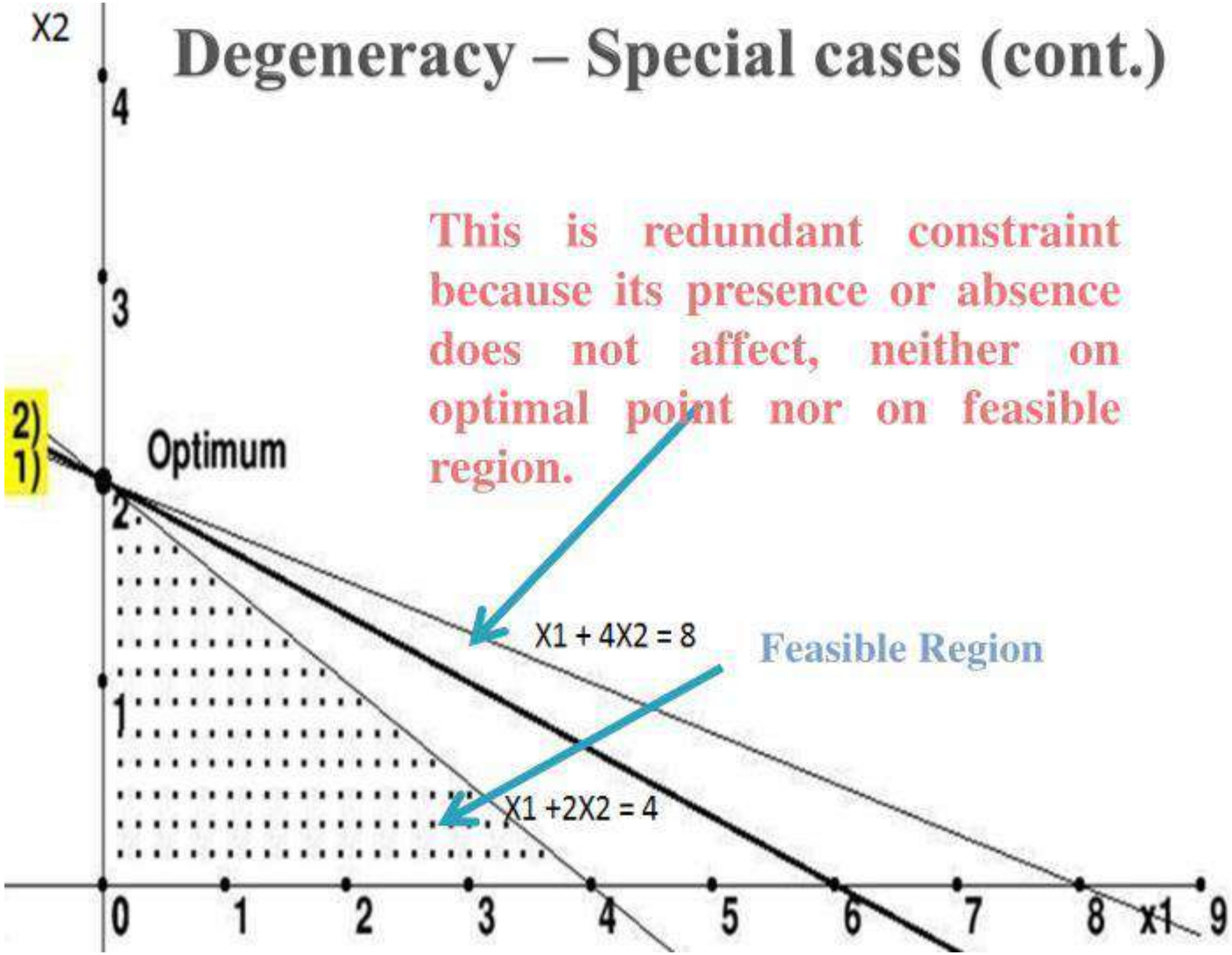
$X_1 + 2x_2 \leq 4$

$X_1, x_2 \geq 0$

Degeneracy

- (0,2)→ Max $Z = 3x_1+9x_2$ = 3*0 + 9 *2 = 18
- (4,0) → 3*4 + 9*0 = 12
- Optimum solution is Z = 18 for $x_1 = 0$ & $x_2 = 2$
- Since one of the variables is zero in the optimum solution, the given LPP has degenerate solution

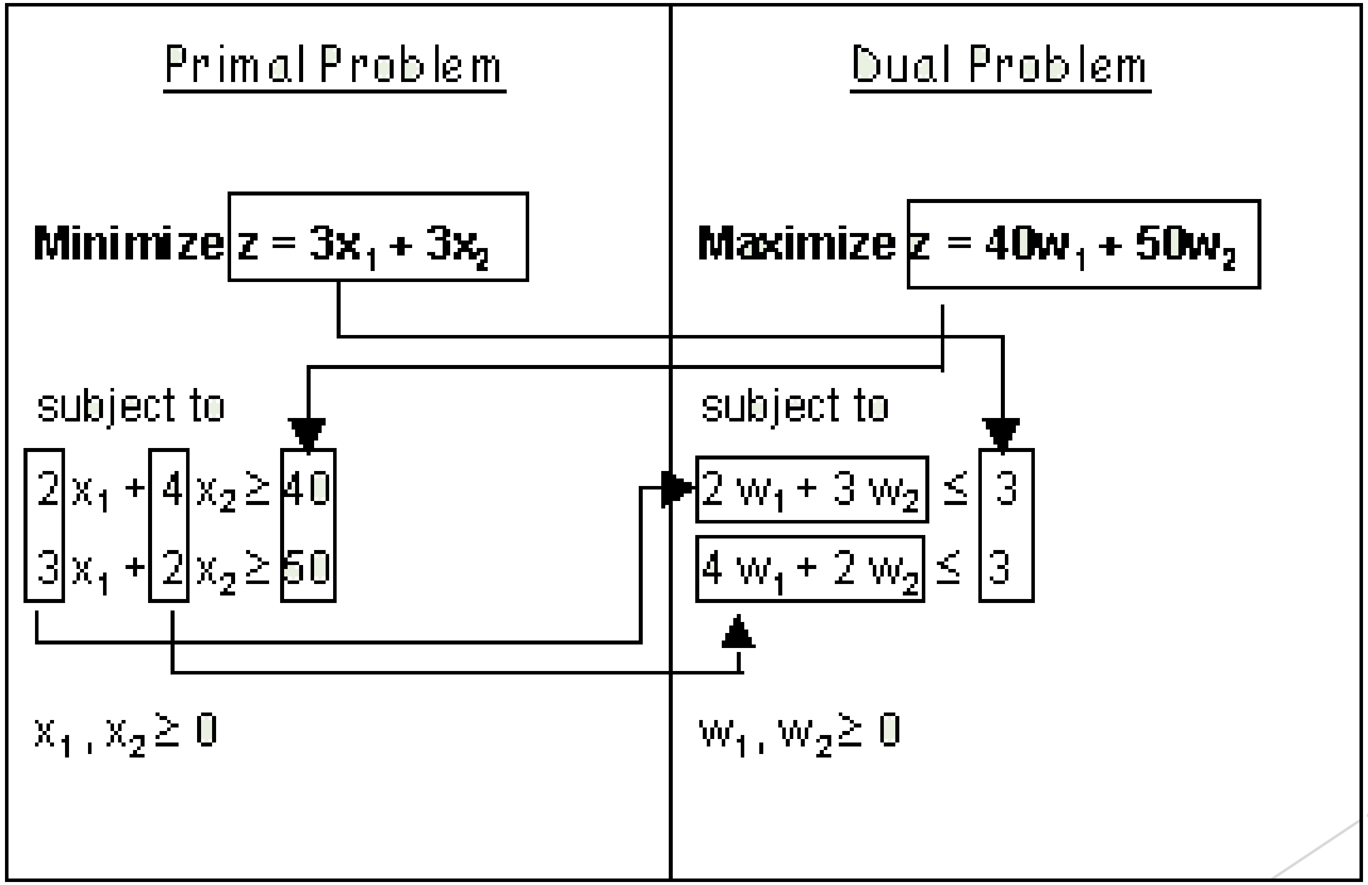
Degeneracy – Special cases (cont.)



Dual LPP

- The **dual** of a given linear program (LP) is another LP that is derived from the original (the primal) LP
- **Duality in Linear Programming** states that every linear programming problem has another linear programming problem related to it and thus can be derived from it.
- The original linear programming problem is called **“Primal,”** while the derived linear problem is called **“Dual”**

Formulating Dual



LP: Dual Formation

- Formulation of the Dual from the Prime:
  - Standardize constraint set:
    - Max problem - all to (≤) type /Min problem, all to (≥).
    - Replace equality constraint with two inequality ones.
  - Transforming standardized Prime to Dual following rules.
    - No. of Variables in Dual = No. of Constraints in Prime
    - No. of Constraints in Dual = No. of Variables in Prime
    - C_j in Prime → b_i in Dual / b_i in Prime → C_j in Dual
    - a_{ij} in Prime → a_{ji} in Dual

Dual of LPP

Primal	Dual
Maximisation form	Minimisation form
No of variables	No of constraints
No of constraints	No of variables
Lesser than or equal to	Greater than or equal to
Equality constraint	Unrestricted variable
Coefficient of variables in objective function	RHS of constraint
RHS of constraint	Coefficient of variables in objective function

Dual LPP

Example

Primal

$$\begin{array}{llll} Max & 40x_1 + 30x_2 & & (profits) \\ s.t. & x_1 + \ \ x_2 \leq 120 & & (land) \\ & 4x_1 + 2x_2 \leq 320 & & (labor) \\ & x_1, \ \ x_2 \geq \ 0 & & \end{array}$$

Dual

$$\begin{array}{llll} & (land) \ \ (labor) & & \\ Min & 120y_1 + 320y_2 & & \\ s.t. & y_1 + \ \ 4y_2 \geq 40 & & (x_1) \\ & y_1 + \ \ 2y_2 \geq 30 & & (x_2) \\ & y_1, \ \ y_2 \geq \ 0 & & \end{array}$$

Formulating Dual LPP

TOYCO primal	TOYCO dual
Maximize $z = 3x_1 + 2x_2 + 5x_3$	Minimize $w = 430y_1 + 460y_2 + 420y_3$
subject to	subject to
$x_1 + 2x_2 + x_3 \leq 430$ (Operation 1)	$y_1 + 3y_2 + y_3 \geq 3$
$3x_1 + 2x_3 \leq 460$ (Operation 2)	$2y_1 + 4y_3 \geq 2$
$x_1 + 4x_2 \leq 420$ (Operation 3)	$y_1 + 2y_2 \geq 5$
$x_1, x_2, x_3 \geq 0$	$y_1, y_2, y_3 \geq 0$
Optimal solution:	Optimal solution:
$x_1 = 0, x_2 = 100, x_3 = 230, z = \$1350$	$y_1 = 1, y_2 = 2, y_3 = 0, w = \$1350$