Probability

MAM BS Unit 4

introduction

Generally, in day-to-day conversation the words, probability, possible, chance, likelihood etc., are commonly used. You may have a rough idea of what is meant by these words. For example, we may come across the statements like: the train may come late today, the chance of winning the cricket match etc. It means there is uncertainity about the happening of the event(s). We live in a world where we are unable to forecast the future with complete certainty. Our need to cope with uncertainty leads us to the study and use of probability. In statistics, the term probability is established by definition and is not related to beliefs.

If the conditions of certainty only were to prevail, life would have been much more simple. As is obvious there are numerous real life situations in which conditions of uncertainty and risk prevail. Consequently, we have to rely on the theory of chance or probability in order to have a better idea about the possible outcomes. There are social, economic and business sectors in which decision making becomes a real challenge for the managers. They may be in the dark about the possible consequences of their decisions and actions. Due to increasing competitiveness the stakes have become higher and cost of making a wrong decision has become enormous.

- **Random Experiment**: A set of activities performed in a homogenous condition repetitively constitutes a random experiment. It results in various possible outcomes. An experiment, therefore, may be a single-trial, two-trial, or n-trial experiment. It may, thus, be noted that an experiment is determined in terms of the nature of trial and the number of times the trial is repeated.

- **Trial and Events**: To conduct an experiment once is termed as trial, while possible outcomes or combination of outcomes is termed as events. For example, toss of a coin is a trial, and the occurrence of either head or a tail is an event.
- **Sample Space**: The set of all possible outcomes of an experiment is called the sample space for that experiment. For example, in a single throw of a dice, the sample space is (1, 2, 3, 4, 5, 6)

Collectively Exhaustive Events: It is the set of all possible events that can result from an experiment. It is obvious that the sum total of probability value of each of these events will always be one. For example, in a single toss of a fair coin, the collectively exhaustive events are either head or tail.

Since $P(H) = 0.5$ and $P(T) = 0.5$

$\therefore P(H) + P(T) = 0.5 + 0.5 = 1.0$

Mutually Exclusive Events: Two events are said to be mutually exclusive events if the occurrence of one event implies no possibility of occurrence of the other event. For example, in throwing an unbiased dice, the occurrence of the number at the top prevents the occurrence of other numbers on it.

Equally Likely Events: When all the possible outcomes of an experiment have an equal probability of occurrence, such events are called equally likely events. For example, in case of throwing of a fair coin, we have already seen that P(Head) = P (Tail) = 0.5

Let us, now, discuss the concepts and approaches to determine and interpret probability. There are two fundamental concepts of probability. They are:

(i) The value of probability of any event lies between 0 to 1. This may be expressed as follows:

$$0 \leq P\ (\text{Event}) \leq 1$$

If the value of probability of an event is equal to zero, then the event is never expected to occur and if the probability value is equal to one, the event is always expected to occur.

(ii) The sum of the simple probabilities for all possible outcomes of an activity must be equal to one.

Before proceeding further, first of all, let us discuss different approaches to defining probability concept.

Approaches to Probability

a) **Classical Approach:** The classical approach to defining probability is based on the premise that all possible outcomes or elementary events of experiment are mutually exclusive and equally likely. The term equally likely means that each of all the possible outcomes has an equal chance of occurance. Hence, as per this approach, the probability of occuring of any event 'E' is given as:

$$P(E) = \frac{\text{No. of outcomes where the event occurs } [n(E)]}{\text{Total no. of all possible outcomes } (n(S)]}$$

This approach is also known as 'A Priori' probability as when we are performing the event using a fair coin, standard card, unbiased dice, then we can tell in advance the probability of happening of some event. We need not perform the actual experiment to find the required probability.

Now we are in a position to understand the classical definition of probability. The definition states:

> If a trial can result in n mutually exclusive, equally likely and exhaustive outcomes and out of which m outcomes are favourable to an event A, the probability of A, denoted by P(A), is then $P(A) = \frac{m}{n}$.

It is clear that if A is an impossible event, that is, none of the n possible outcomes favours the occurrence of the event A, we have $m = 0$. The probability of A in that case is $P(A) = \frac{m}{n} = \frac{0}{n} = 0$

On the other hand, if A is a certain event, that is, all of the n possible outcomes favour the occurrence of the event A, we have $m = n$. The probability of A in that case is $P(A) = \frac{m}{n} = \frac{n}{n} = 1$

Example: A bag contains 17 balls numbered 1,2,…17. one ball is taken out at random from this bag.

Let ‘A’ be the event of taking out a ball with number that is multiple of ‘3’.

Let ‘B’ be the event of taking out a ball with number that is multiple of ‘5’.

Favorable numbers for event A = 3,6,9,12,15

P (A) = 5/17

Favorable numbers for event B =5,10,15

P (B) = 3/17

Example 13.2 A fair dice is thrown. What is the probability that either 1 or 6 will show up?

A dice has six faces with 1, 2, 3, 4, 5 and 6 dots printed on them and any one of these faces will show up when the dice is thrown. Thus the number of exhaustive outcomes $n = 6$. Now, the face with 1 dot favours the required event and the face with 6 dots also satisfies the required event. So, the number of outcomes favouring the required event is $m = 2$. If $P(1\ or\ 6)$ is the probability of either 1 or 6 then

$$P(1\ or\ 6) = \frac{m}{n} = \frac{2}{6} = \frac{1}{3}$$

Example: When we toss a fair coin, the probability of getting a head would be:

$$P(\text{Head}) = \frac{\text{Total number of favourable outcomes}}{\text{Total no. of all possible outcomes}} = \frac{1}{1+1} = \frac{1}{2}$$

Similarly, when a dice is thrown, the probability of getting an odd number is $\frac{3}{6}$ or $\frac{1}{2}$.

The premise that all outcomes are equally likely assumes that the outcomes are symmetrical. Symmetrical outcomes are possible when a coin or a die being tossed are fair. This requirement restricts the application of probability only to such experiments which give rise to symmetrical outcomes. The classical approach, therefore, provides no answer to problems involving asymmetrical outcomes. And we do come across such situations more often in real life.

Limitations of the Classical Definition

The classical definition has some serious drawbacks. They are:

a) The classical definition can be applied only if various outcomes of the trials are equally likely or equally probable. But in practice the outcomes need not be always equally likely. For example, if a coin is biased in favour of head, the classical definition fails to give the probability of a head or a tail.

b) The classical definition is valid for a finite number of outcomes of a trial. It fails when the number of outcomes becomes infinity. In fact, even in the case of a finite number of outcomes, it may not be practically feasible to enumerate all the cases.

Exercise

1) A box contains 4 white balls and 6 red balls. A ball is drawn without looking into the box. What is the probability that it is a white ball?

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2) A six-faced dice is thrown. What is the probability of getting an even number?

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3) A coin is tossed twice. What is the probability of getting either two heads or two tails?

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4) A card is drawn from a pack of 52 cards. What is the probability of not getting a king?

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13.2.2 Relative Frequency or Statistical Definition

Another definition that has often been used is the relative frequency definition of probability. If we repeat a trial and observe the occurrence of an event, we shall see that as the number of trials is progressively increased, the ratio of the number of times a particular event occurs to the total number of trials tends to stabilise at a particular value. Now, the number of times an event occurs is its frequency and when this frequency is divided by the total number of trials, we get the relative frequency of the event. Thus in other words, when the number of trials becomes sufficiently large, the relative frequency of an event tends to a limit. According to the relative frequency definition, this limiting value is the probability of the event under consideration. Suppose, we repeat the experiment of tossing a coin and observe the number of times head occurs. We shall find that as we increase the number of tosses from say, 10 to 100 to 1000 to 10000 and so on, the relative frequency of head will gradually stabilise at $\frac{1}{2}$. Thus, the probability of head in the toss of a fair coin is $\frac{1}{2}$.

Mathematically, if n is the total number of trials out of which, an event A occurs m times, the probability of A

$$P(A) = \lim_{n \to \infty} \frac{m}{n}$$

Subjective approach

According to this approach probability is based on the experience and the judgment of the person making this estimate. This may differ from person to person, depending on one’s perception of the situation and the past experience. Subjective probability can be defined as based on the available evidence. Sometimes logic and past data are not so useful in determining the probability value, in those cases the subjective approach of the assessor is being used to find that probability. This approach is so flexible that it may be applied in a number of situations where the earlier two approaches fail to offer a satisfactory answer.

Notations

Before considering various probability laws, let us be familiar with certain notations.

a) If A and B are two events, then $P(A \cup B)$ or $P(A+B)$ denotes the probability that either A occurs or B occurs or both occur simultaneously. It can also be interpreted as the probability of the occurrence of at least one of the two events A and B. The symbol $\cup$ above represents 'union' between two events. (Read $A \cup B$ as 'A union B').

b) $P(A \cap B)$ or $P(AB)$ denotes the probability of the simultaneous occurrence of both A and B. (Read $A \cap B$ as 'A intersection B').

c) $P(A / B)$ denotes the *conditional probability* of the occurrence of A given that B has already occurred.

Probability Theorems

The following rules of probability are useful for calculating the probability of an event/events under different situations.

13.5.1 Addition Rule for Mutually Exclusive Events

If two events, A and B, are mutually exclusive, then the probability of occurance of either A or B is given by the following formula:

$$P(A \text{ or } B) = P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

If A and B are mutually exclusive events, this rule is depicted in Figure 13.1, below.

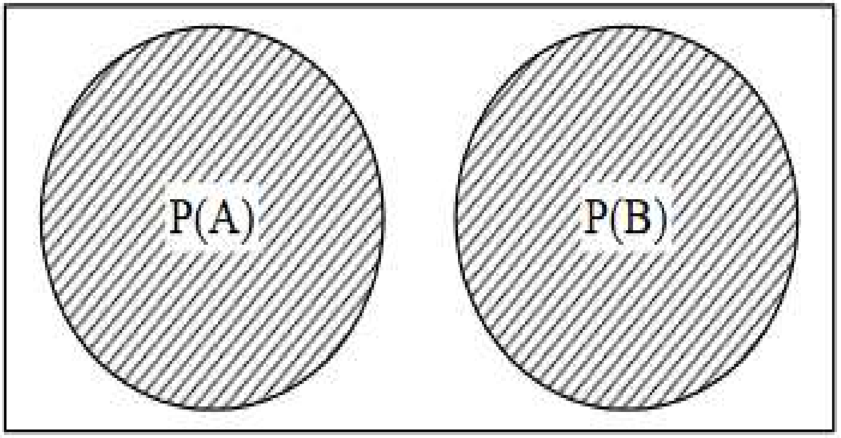

Figure: 13.1

The essential requirement for any two events to be mutually exclusive is that there are no outcomes common to the occurance of both. This condition is satisfied when sample space does not contain any outcome favourable to the occurance of both A and B means $A \cap B = \varphi$

Illustration 1: In a game of cards, where a pack contains 52 cards, 4 categories exist namely spade, club, diamond, and heart. If you are asked to draw a card from this pack, what is the probability that the card drawn belongs to either spade or club category.

Solution: Here, P (Spade or club) = P (Spades) + P (Club)

$$\text{Where P (Spade)} = \frac{13}{52} = \frac{1}{4} \text{ and P (Club)} = \frac{13}{52} = \frac{1}{4}$$

$$\therefore \text{P (Spade or Club)} = \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$$

There is an important special case for any event E, either E happens or it does not. So, the events E and not E are exhaustive and exclusive.

So, P (E) + P (not E) = 1

or, P (E) = 1–P (not E)

Sometimes P (not E) is also written as either $P(E^{|}) = 1 - P(\overline{E})$

So, $P(E) = 1 - P(E^{|}) = 1 - P(\overline{E})$.

13.5.2 Addition Rule for Non-Mutually Exclusive Events

Non-mutually exclusive (overlapping) events present another significant variant of the additive rule. Two events (A and B) are not mutually exclusive if they have some outcomes common to the occurance of both, then the above rule has to be modified in order to account for the overlapping areas, as it is clear from Figure 13.2. below.

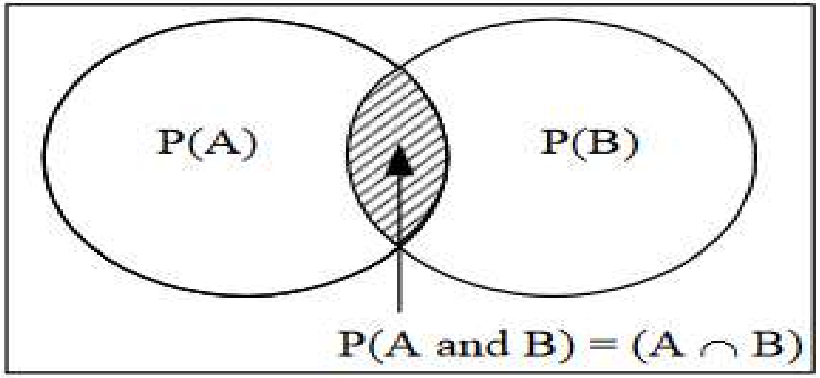

Figure: 13.2

In this situation, the probability of occurrence of event A or event B is given by the formula

$$P\ (A \text{ or } B) = P\ (A \cup B) = P\ (A) + P\ (B) - P\ (A \text{ and } B)$$

where P (A and B) is the joint probability of events A and B, i.e., both occuring together and is usually written as P (A $\cap$ B).

Thus, it is clear that the probability of outcomes that are common to both the events is to be subtracted from the sum of their simple probability.

Consider the following illustrations to understand the application of this concept.

Illustration 2: The event of drawing either a Jack or a spade from a well-shuffled deck of playing cards. Find the probability.

Solution: These events are not mutually exclusive, so the required probability of drawing a Jack or a spade is given by:

P (Jack or Spade) = P (Jack) + P (Spade) – P (Jack and Spade)

$$= \frac{4}{52} + \frac{13}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Probability under statistical independence

The two or more events are termed as statistically independent events, if the occurrence of any one event does not have any effect on the occurrence of any other event. For example, if a fair coin is tossed once and suppose head comes, then this event has no effect in any way on the outcome of second toss of that same coin. Similarly, the results obtained by drawing hearts from a pack has no effect in any way on the results obtained by throwing a dice. These events thus are being termed as statistically independent events. There are three trypes of probability under statistically independent case.

a) Marginal Probability;

b) Joint Probability;

c) Conditional Probability.

a) Marginal Probability Under Statistical Independence

A Marginal/Simple/Unconditional probability is the probability of the occurrence of an event. For example, in a fair coin toss, probability of having a head is:

$$P(H) = \tfrac{1}{2} = 0.5$$

Therefore, the marginal probability of an event (i.e. having a head) is 0.5. Since, the subsequent tosses are independent of each other, therefore, it is a case of statistical independence.

Another example can be given in a throw of a fair die, the marginal probability of the face bearing number 3, is:

$$P(3) = 1/6 = 0.166$$

Since, the tosses of the die are independent of each other, this is a case of statistical independence.

b) **Joint Probability Under Statistical Independence**

This is also termed as "**Multiplication Rule of Probability**". In many situations we are interested in finding out the probability of two or more events either occurring together or in quick succession to each other, for this purpose the concept of joint probability is used.

This joint probability of two or more statistically independent events occurring together is determined by the product of their marginal probability. The corresponding formula may be expressed as:

$$P\ (A \text{ and } B) = P\ (A) \times P\ (B)$$

Similarly, it can be extended to more than two events also as:

$$P\ (A \text{ and } B \text{ and } C) = P\ (A) \times P\ (B) \times P\ (C) \text{ and so on.}$$

i.e. $$P\ (A \text{ and } B \text{ and } C \text{ and } \ldots) = P\ (A) \times P\ (B) \times P\ (C) \times \ldots$$

For instance, when a fair coin is tossed twice in quick succession, the probability of head occuring in both the tosses is:

$$P(H_1 \text{ and } H_2) = P\,(H_1) \times P\,(H_2)$$
$$= 0.5 \times 0.5 = 0.25$$

Where, H_1 is the occurrence of head in 1st toss, and H_2 is the occurrence of head in 2nd toss.

Take another example: When a fair die is thrown twice in quick succession, then to find the probability of having 2 in the 1st throw and 4 in second throw is, given as:

$$P\,(2 \text{ in } 1^{st} \text{ throw and } 4 \text{ in } 2^{nd} \text{ throw})$$
$$= P\,(2 \text{ in the } 1^{st} \text{ throw}) \times P\,(4 \text{ in the } 2^{nd} \text{ throw})$$
$$= \frac{1}{6} \times \frac{1}{6} = \frac{1}{36} = 0.028$$

c) Conditional Probability Under the Condition of Statistical Independence

The third type of probability under the condition of statistical independence is the Conditional Probability. It is symbolically written as P (A/B), i.e., the conditional probability of occurrence of event A, on the condition that event B has already occurred.

In case of statistical independence, the conditional probability of any event is akin to its marginal probability, when both the events are independent of each other.

Therefore, $P(A/B) = P(A)$, and

$$P(B/A) = P(B).$$

For example, if we want to find out what is the probaility of heads coming up in the second toss of a fair coin, given that the first toss has already resulted in head. Symbolically, we can write it as:

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$$P\ (H_2/H_1)$$

As, the two tosses are statistically independent of each other

so, $P\ (H_2/H_1) = P\ (H_2)$

The following table 13.1 summarizes these three types of probabilities, their symbols and their mathematical formulae under statistical independence.

Table 13.1

Probability's Type	Symbol	Formula
Marginal	P (A)	P (A)
Joint	P (AB)	P (A) × P (B)
Conditional	P (B/A)	P (B)

13.3.2 Multiplication Law

This law states that the probability of the simultaneous occurrence of the two events A and B is equal to the product of

i) the probability of A and the conditional probability of B given that A has already occurred

or

ii) the probability of B and the conditional probability of A given that B has already occurred.

In symbols,

$$P(A \cap B) = P(A) \cdot P(B/A) = P(B) \cdot P(A/B)$$

Using the multiplication law, we can find the *conditional probabilities*

$$P(B/A) = \frac{P(A \cap B)}{P(A)}$$

and

$$P(A/B) = \frac{P(A \cap B)}{P(B)}$$

In the case of multiplication law, certain modification is required for *independent events*.

Suppose the occurrence of B does not depend upon the occurrence of A and vice versa, then the two events A and B are said to be mutually independent. In this case the two conditional probabilities $P(B/A)$ and $P(A/B)$ are equal to their respective non-conditional simple probabilities. Hence, for independence

$$P(B/A) = P(B) \qquad \text{and}$$

$$P(A/B) = P(A)$$

Thus, for two independent events A and B, the probability of their simultaneous occurrence is the product of their respective probabilities.

$$P(AB) = P(A) \cdot P(B) = P(B) \cdot P(A)$$

Example 13.7 A dice is thrown. What is the probability of getting a number less than 5 or an odd number?

Let A be the event of a number less than 5 and B be the event of an odd number. We should note here that the two events are not mutually exclusive as a number can be both less than 5 and an odd number. So the required probability is obtained by applying the formula

$$P(A \cap B) = P(A) + P(B) - P(A \cap B)$$

Now in a dice, out of 6 numbers, there are 4 numbers (1, 2, 3, and 4) less than 5. Therefore,

$$P(A) = \frac{4}{6} = \frac{2}{3}.$$

Again, there are 3 odd numbers (1,3 and 5) out of the possible six numbers. So

$$P(B) = \frac{3}{6} = \frac{1}{2}$$

Suppose $P(B/A)$ is the probability of an odd number given that it is less than five. Then

$$P(B/A) = \frac{2}{4} = \frac{1}{2}$$

Now

$$P(AB) = P(A) \cdot P(B/A) = \frac{2}{3} \times \frac{1}{2} = \frac{1}{3}$$

Thus, the probability of getting a number less than 5 or an odd number is

$$\frac{2}{3} + \frac{1}{2} - \frac{1}{3} = \frac{5}{6}$$

Exercise

1) A student takes Mathematics and English tests. His independent chances of passing the two tests are $\frac{2}{3}$ and $\frac{3}{4}$ respectively. What is the probability that

   a) he passes at least one test?

   b) he fails in both the tests?

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Examples

1. Contractor 'X' has submitted tender for two contracts, A & B. The probability of getting contract A is $\frac{1}{4}$ and the contract B is $\frac{1}{2}$ and both the contracts is $\frac{1}{8}$. Find the probability that X will get contract A or B.

Ans: Use Addition rule of Probability

$$P(A \text{ or } B) = P(A) + P(B) - P(A\&B)$$

$$= \frac{1}{4} + \frac{1}{2} - \frac{1}{8} = \frac{5}{8} = 0.625$$

2. A firm has invited applications for Manager post. The probability that an applicant has post graduate qualification is 0.30 and that he has work experience is 0.70, and he has both is 0.40. Assuming that 50 persons have applied for this post, find out how many applicants would have either post graduate qualification or work experience.

Sol: let A be the event of having post graduate qualification

let B be the event of having work experience

Use Addition rule of Probability

$$P(A \text{ or } B) = P(A) + P(B) - P(A\&B)$$

$$= 0.30 + 0.70 - 0.40 = 0.60$$

Since the number of applicants are 50, the number having post graduate qualification and work experience is 0.60 x 50 = 30

3. Suppose we have a box with 3 red, 2 black and 5 white balls. Each time a ball is drawn, it is returned to the box. What is the probability of drawing:

(a) either a red or black ball

(b) either a white or black ball

Sol: P(Red) = 3/10 = 0.30

P(Black)= 2/10 = 0.20

P(White) = 5/10 = 0.50

Because the events are mutually exclusive, we have

P(Red or Black) = P(Red) + P(Black) = 0.30+0.20 =0.50

P(White or Black) = P(White) + P(Black) = 0.20 + 0.50 = 0.70

4. In a class of 100 students, 20 students got A grade, 25 got B grade, 20 got C grade and the rest got D grade. What is the probability of selecting a student who has

(a) either A grade or B grade

(b) either C grade or D grade

P (A grade) = 0.20   P(B grade) = 0.25

P(C grade) = 0.20   P(D grade) = 0.35

since they are mutually exclusive events, we have

P(A or B grade) = 0.20 + 0.25 = 0.45

P(C or D grade) = 0.20 + 0.35 = 0.55